hence we obtain at once $p+r$, and $q+s$, which will render the solution of the equations easy, at all events, in all possible cases.

I would moreover remark that the validity of the process depends on $\sqrt{(p+qx+rx^2+sx^3)}$ expanding in a converging series, so that the method of evaluation here, α of course in (109), (110), depends on certain conditions, to which the constants in the integrals must be subject.

II. "On certain Definite Integrals." No. 7. By W. H. L. Russell, F.R.S. Received January 6, 1880.

By a development of the methods indicated in the former papers we obtain the following integrals:—

$$(115.)\quad \int_0^{\frac{\pi}{2}} d\theta\,.\,\frac{\cos^n\theta\cos(n-2)\theta+\alpha\cos^{n+1}\theta\cos(n-3)\theta}{\sin^2\theta+(1+\alpha)^2\cos^2\theta} =\frac{\pi}{2^n}\left\{\frac{n}{\alpha+2}-\frac{\alpha}{(\alpha+2)^2}\right\}.$$

$$(116.)\quad \int_0^{\frac{\pi}{2}} d\theta\cos^2\theta\,\epsilon^{2x\cos^2\theta}\cos(x\sin 2\theta)=\frac{\pi}{8}(x+2)\,\epsilon^{x}.$$

$$(117.)\quad \int_0^{\frac{\pi}{2}} d\theta\,\frac{\cos^n\theta\cos(n-2)\theta+\alpha^3\cos^{n+3}\theta\cos(n-5)\theta}{1+2\alpha^3\cos^3\theta\cos 3\theta+\alpha^6\cos^6\theta} =\frac{\pi}{2^{n-2}}\left\{\frac{n}{\alpha^3+8}-\frac{3\alpha^3}{(a^3+8)^2}\right\}$$

$$(118.)\quad \int_0^{\frac{\pi}{2}} d\theta\cos^2\theta\,\epsilon^{\alpha\cos^3\theta\cos 3\theta}\cos(\alpha\cos^3\theta\sin 3\theta)=\frac{\pi}{8}\left(2+\frac{3\alpha}{8}\right)\epsilon^{\frac{\alpha}{8}}.$$

$$(119.)\quad \int_0^{\frac{\pi}{2}} \frac{\cos^\mu\theta\cos\mu\theta+\alpha\cos^{\mu+1}\theta\cos(\mu-1)\theta}{\sin^2\theta+(\alpha+1)^2\cos^2\theta}d\theta=\frac{\pi}{2^\mu}\frac{1}{\alpha+2}.$$

$$(120.)\quad \int_0^{\frac{\pi}{2}} d\theta\,\frac{\cos^\mu\theta\cos\mu\theta+(\alpha+\beta)\cos^{\mu+\theta}\cos(\mu-1)\theta+\alpha\beta\cos^{\mu+1}\theta\cos(\mu-2)\theta}{(\sin^2\theta+(\alpha+1)^2\cos^2\theta)(\sin^2\theta+(\beta+1)^2\cos^2\theta} =\frac{\pi}{2^{\mu-1}(\alpha+2)(\beta+2)}.$$

This integral may be written

$$(121.)\quad \int_0^{\frac{\pi}{2}} d\theta\,\frac{\cos^\mu\theta\cos\mu\theta+p\cos^{\mu+1}\theta\cos(\mu-1)\theta+q\cos^{\mu+2}\theta\cos(\mu-2)\theta}{\sin^4\theta+(p^2+2p-2q+2)\sin^2\theta\cos^2\theta+(p+q+1)^2\cos^4\theta} =\frac{\pi}{2^{\mu-1}(q+2p+4)}$$

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(122.) $$\int_0^{\frac{\pi}{2}} d\theta\, \epsilon^{\alpha\cos^2\theta} \frac{\cos^{\mu-1}\theta}{\sin\theta} \sin(\alpha\sin\theta\cos\theta+\mu\theta)=\frac{\pi\epsilon^{\alpha}}{2}.$$

(123.) $$\int_0^{\frac{\pi}{2}} d\theta\cos\theta\frac{\mu\cos\theta(\beta+\cos\tan\theta)-\lambda\sin\theta\sin\tan\theta}{(1+2\beta\cos\tan\theta+\beta^2)(\lambda^2\sin^2\theta+\mu^2\cos^2\theta)}=\frac{\pi}{2(\lambda+\mu)(\epsilon+\beta}.$$

(124.) $$\int_0^{\pi} \frac{\theta d\theta(1-x^2\cos^2\theta)\sin\theta}{(1+x^2\cos^2\theta)\sqrt{1+x^4\cos^4\theta}}=\frac{\pi}{x\sqrt{2}}\,.\,\sin^{-1}\frac{x\sqrt{2}}{1+x^2}.$$

(125.) $$\int_0^{\pi} d\theta\theta\sin\theta\frac{x^{2n}\cos^{2n}\theta}{(a^2+x^2\cos^2\theta)\,..\,(e^2+x^2\cos^2\theta)}=\frac{\pi}{x}\int\frac{dx\,.\,x^{2n}}{(a^2+x^2)\,..\,(e+x^2)}.$$

(126.) $$\int_0^{\pi} \frac{\cos r\theta}{(1-2\alpha\cos\theta+\alpha^2)(1-2\beta\cos\theta+\beta^2)}d\theta$$
$$=\frac{\pi}{(1-\alpha\beta)(\alpha-\beta)}\left\{\frac{\alpha^{r+1}}{1-\alpha^2}-\frac{\beta^{r+1}}{1-\beta^2}\right\}.$$

(127.) Hence we see the values of

$$\int_0^{\pi} \frac{d\theta\,.\,(f\epsilon^{\theta i}+f\epsilon^{-\theta i})}{(1-2\alpha\cos\theta+\alpha^2)(1-2\beta\cos\theta+\beta^2)}.$$

By a similar method we may find

(128.) $$\int_0^{\pi} \frac{d\theta\cos r\theta}{(1-2a\cos\theta+\alpha^2)(1-2\beta\cos\theta+\beta^2)\,\ldots\,(1-2\mu\cos\theta+\mu^2)}.$$

(129.) $$\int_0^{\pi} \frac{d\theta\,.\,\sin(2r+1)\theta}{(\lambda^2\cos^2\theta+\mu^2\sin^2\theta)\sin\theta}=\frac{\pi}{\lambda^2}+\frac{\pi(\mu-\lambda)}{\lambda^2\mu}\left(\frac{\mu-\lambda}{\mu+\lambda}\right)^r.$$

(130.) $$\int_0^{\pi} \frac{d\theta\,.\,\epsilon^{\alpha\cos 2\theta}\sin(\alpha\sin 2\theta+\theta)}{(\lambda^2\cos^2\theta+\mu^2\sin^2\theta)\,.\,\sin\theta}=\frac{\pi}{\lambda^2}\,\epsilon^{\alpha}+\frac{\pi(\mu-\lambda)}{\lambda^2\mu}\,\epsilon^{\alpha\cdot\frac{\mu-\lambda}{\mu+\lambda}}.$$

(131.) We may also find

$$\int_0^{\pi} \frac{d\theta\sin(2r+1)\theta}{\sin\theta(\lambda_1{}^2\sin^2\theta+\mu_1{}^2\cos^2\theta)(\lambda_2{}^2\sin^2\theta+\mu_2{}^2\cos^2\theta)\,..\,(\lambda_n{}^2\sin^2\theta+\lambda_n\cos^2\beta)}.$$

(132.) $$\int_0^{\pi} \frac{\sin r\theta\,.\,d\theta}{1-2\alpha\cos\theta+\alpha^2}=\frac{\pi\alpha^r}{(1-\alpha^2)}-\frac{1}{(1-\alpha^2)}\,.\left(\alpha^r-\frac{1}{\alpha^r}\right)\log_{\epsilon}\frac{1+\alpha}{1-\alpha}$$
$$+\frac{2}{1-\alpha^2}\left\{\left(\alpha^{r-1}-\frac{1}{\alpha^{r-2}}\right)+\frac{1}{3}\left(\alpha^{r-3}-\frac{1}{\alpha^{r-3}}\right)+\ldots\frac{1}{r-1}\left(\alpha-\frac{1}{\alpha}\right)\right\},$$

when (r) is even, with a similar expression when (r) is odd. I shall now hope to prove that every function of an algebraical magnitude may be regarded as a centre, from which systems of definite integrals emanate in all directions, like rays from a star, in such a manner, that the value of each integral is equivalent to the original function transformed by a known symbol.